

The Wigner-Yanase information can increase under phase sensitive incoherent operations

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We found that the Wigner-Yanase skew information, which has been recently proposed as a measure of coherence in [Phys. Rev. Lett. **113**, 170401(2014)], can increase under a class of operations which may be interpreted as incoherent following the framework of Baumgratz et al., while being phase sensitive.

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Introduction—Coherence arising from quantum superposition plays a central role for quantum mechanics. Quantum coherence is an important subject in quantum theory and quantum information science which is a common necessary condition for both entanglement and other types of quantum correlations. It is the key resource for quantum technology, with applications in quantum optics, information processing, metrology and cryptography. Up to now, several themes of coherence have been considered such as witness coherence [2], catalytic coherence [3], the thermodynamics quantum coherence [4], and the role of coherence in biological system [5]. But, given a quantum state, how much the coherence does it contain? How to quantify the quantum coherence? There is no well-accepted efficient method to quantify the coherence in quantum system until recently. T. Baumgratz etc. have introduced a rigorous framework for the quantification of quantum coherence [6]. Let \mathcal{H} be a finite dimensional Hilbert space with $d = \dim(\mathcal{H})$. Fixing a basis $\{|i\rangle\}_{i=1}^d$, we call all density operators (quantum states) that are diagonal in this basis incoherent, and this set of quantum states will be labelled by \mathcal{I} . Quantum operations are specified by a finite set of Kraus operators $\{A_n\}$ satisfying $\sum_n A_n^\dagger A_n = I$, I is the identity operator on \mathcal{H} . From [6], quantum operations are incoherent if they fulfil $A_n \rho A_n^\dagger / \text{Tr}(A_n \rho A_n^\dagger) \in \mathcal{I}$ for all $\rho \in \mathcal{I}$ and for all n . Based on Baumgratz's suggestion [6], any proper measure of coherence \mathcal{C} must satisfy the following axiomatic postulates.

- (i) The coherence vanishes on the set of incoherent states (faithful criterion), $\mathcal{C}(\rho) = 0$ for all $\rho \in \mathcal{I}$;
- (ii) Monotonicity under incoherent operations Φ , $\mathcal{C}(\Phi(\rho)) \leq \mathcal{C}(\rho)$;
- (iii) Non-increasing under mixing of quantum states (convexity),

$$\mathcal{C}\left(\sum_n p_n \rho_n\right) \leq \sum_n p_n \mathcal{C}(\rho_n)$$

for any ensemble $\{p_n, \rho_n\}$.

For every self-adjoint non-degenerate $d \times d$ diagonal matrix K , it was shown that the Wigner-Yanase skew information

$$\mathcal{C}(\rho, K) = -\frac{1}{2} \text{Tr}([\sqrt{\rho}, K]^2)$$

satisfies the above three postulates and so is a measure of the K -coherence of the state ρ [1], where $[\sqrt{\rho}, K] = \sqrt{\rho}K - K\sqrt{\rho}$. However, the following example shows that the Wigner-Yanase skew information can increase under an incoherent operation.

Assume $\dim(\mathcal{H}) = 3$. Let

$$\begin{aligned} K &= |1\rangle\langle 1| + 10|2\rangle\langle 2| + 5|3\rangle\langle 3|, \\ |\psi\rangle &= \frac{1}{\sqrt{3}}|1\rangle + \frac{1}{\sqrt{3}}|2\rangle + \frac{1}{\sqrt{3}}|3\rangle, \\ |\phi\rangle &= \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle, \\ A_1 &= \frac{1}{\sqrt{2}}|1\rangle\langle 1| + \frac{1}{\sqrt{2}}|2\rangle\langle 2|, \\ A_2 &= \frac{1}{\sqrt{2}}|1\rangle\langle 2| + \frac{1}{\sqrt{2}}|2\rangle\langle 3|, \\ A_3 &= \frac{1}{\sqrt{2}}|1\rangle\langle 3| + \frac{1}{\sqrt{2}}|2\rangle\langle 1|. \end{aligned}$$

It is easy to see that $\sum_{n=1}^3 A_n^\dagger A_n = I$. Furthermore, for any diagonal density operator $\rho \in \mathcal{I}$, we have $A_n \rho A_n^\dagger / \text{Tr}(A_n \rho A_n^\dagger) \in \mathcal{I}$ for all n . That is to say, the Kraus operators $\{A_n\}_{n=1}^3$ define an incoherent operation. By a direct computation, one can get $A_n |\psi\rangle = \frac{1}{\sqrt{3}} |\phi\rangle$, $n = 1, 2, 3$. Hence

$$\sum_{n=1}^3 A_n |\psi\rangle\langle \psi| A_n^\dagger = |\phi\rangle\langle \phi|.$$

However, it is easy to check that

$$\mathcal{C}(|\phi\rangle\langle \phi|, K) = \frac{81}{4} > \mathcal{C}(|\psi\rangle\langle \psi|, K) = \frac{122}{9}.$$

This shows that Wigner-Yanase skew information $\mathcal{C}(\cdot, K)$ can increase under incoherent operations.

In this paper, it will be proved that the Wigner-Yanase skew information can increase under a class of phase sensitive incoherent operations.

Result—For any self-adjoint non-degenerate $d \times d$ ($d \geq 3$) diagonal matrix K , $\mathcal{C}(\cdot, K)$ can increase under a class of phase sensitive incoherent operations.

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Proof. The main idea is to pick a representative of a peculiar class of operations, incoherent but phase sensitive and show that the skew information increases. We firstly treat the case $d = 3$. Because K is non-degenerate, we have $\lambda_i \neq 0$ and $\lambda_i \neq \lambda_j, i, j = 1, 2, 3$. The proof is divided into several cases.

Case i. $\lambda_1 < \lambda_3 < \lambda_2$.

Let

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{3}}|1\rangle + \frac{1}{\sqrt{3}}|2\rangle + \frac{1}{\sqrt{3}}|3\rangle, \\ |\phi\rangle &= \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{3}}|2\rangle + \frac{1}{\sqrt{6}}|3\rangle, \\ A_1 &= \frac{1}{\sqrt{2}}|1\rangle\langle 1| + \frac{1}{\sqrt{3}}|2\rangle\langle 2| + \frac{1}{\sqrt{6}}|3\rangle\langle 3|, \\ A_2 &= \frac{1}{\sqrt{2}}|1\rangle\langle 2| + \frac{1}{\sqrt{3}}|2\rangle\langle 3| + \frac{1}{\sqrt{6}}|3\rangle\langle 1|, \\ A_3 &= \frac{1}{\sqrt{2}}|1\rangle\langle 3| + \frac{1}{\sqrt{3}}|2\rangle\langle 1| + \frac{1}{\sqrt{6}}|3\rangle\langle 2|. \end{aligned}$$

By a direct computation, we have $\sum_{n=1}^3 A_n^\dagger A_n = I$. Moreover, for any diagonal density operator $\rho \in \mathcal{I}$, $A_n \rho A_n^\dagger / \text{Tr}(A_n \rho A_n^\dagger) \in \mathcal{I}$ for all n . That is, the Kraus operators $\{A_n\}_{n=1}^3$ define an incoherent operation. It is easy to check that $A_n|\psi\rangle = \frac{1}{\sqrt{3}}|\phi\rangle, n = 1, 2, 3$. Therefore

$$\sum_{n=1}^3 A_n|\psi\rangle\langle\psi|A_n^\dagger = |\phi\rangle\langle\phi|.$$

After simple computation, we obtain

$$\begin{aligned} \mathcal{C}(|\phi\rangle\langle\phi|, K) - \mathcal{C}(|\psi\rangle\langle\psi|, K) \\ = \frac{(\lambda_3 - \lambda_1)(3\lambda_2 - 3\lambda_3 + \lambda_2 - \lambda_1)}{36} > 0. \end{aligned}$$

Therefore Wigner-Yanase skew information $\mathcal{C}(\cdot, K)$ increases under incoherent operations.

Case ii. $\lambda_2 < \lambda_3 < \lambda_1$.

Let

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{3}}|1\rangle + \frac{1}{\sqrt{3}}|2\rangle + \frac{1}{\sqrt{3}}|3\rangle, \\ |\phi\rangle &= \frac{1}{\sqrt{3}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle + \frac{1}{\sqrt{6}}|3\rangle, \\ A_1 &= \frac{1}{\sqrt{3}}|1\rangle\langle 1| + \frac{1}{\sqrt{2}}|2\rangle\langle 2| + \frac{1}{\sqrt{6}}|3\rangle\langle 3|, \\ A_2 &= \frac{1}{\sqrt{3}}|1\rangle\langle 2| + \frac{1}{\sqrt{2}}|2\rangle\langle 3| + \frac{1}{\sqrt{6}}|3\rangle\langle 1|, \\ A_3 &= \frac{1}{\sqrt{3}}|1\rangle\langle 3| + \frac{1}{\sqrt{2}}|2\rangle\langle 1| + \frac{1}{\sqrt{6}}|3\rangle\langle 2|. \end{aligned}$$

It is easy to see that $\sum_{n=1}^3 A_n^\dagger A_n = I$. After simple computation, for any diagonal density operator $\rho \in \mathcal{I}$, we have $A_n \rho A_n^\dagger / \text{Tr}(A_n \rho A_n^\dagger) \in \mathcal{I}$ for all n . That is to say, the Kraus operators $\{A_n\}_{n=1}^3$ define an incoherent operation. By a direct computation, $A_n|\psi\rangle = \frac{1}{\sqrt{3}}|\phi\rangle, n = 1, 2, 3$. Hence

$$\sum_{n=1}^3 A_n|\psi\rangle\langle\psi|A_n^\dagger = |\phi\rangle\langle\phi|.$$

However, it is easy to check

$$\begin{aligned} \mathcal{C}(|\phi\rangle\langle\phi|, K) - \mathcal{C}(|\psi\rangle\langle\psi|, K) \\ = \frac{(\lambda_3 - \lambda_2)(3\lambda_1 - 3\lambda_3 + \lambda_1 - \lambda_2)}{36} > 0. \end{aligned}$$

This tells us that Wigner-Yanase skew information $\mathcal{C}(\cdot, K)$ increases.

For the remained cases, in order to construct suitable pure state $|\phi\rangle$, one only need to adjust the position of $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}$ according to the order relation of $\lambda_1, \lambda_2, \lambda_3$. For instance, in the case of $\lambda_3 < \lambda_1 < \lambda_2$, we choose

$$|\phi\rangle = \frac{1}{\sqrt{6}}|1\rangle + \frac{1}{\sqrt{3}}|2\rangle + \frac{1}{\sqrt{2}}|3\rangle.$$

The incoherent Kraus operators are constructed as follows,

$$\begin{aligned} A_1 &= \frac{1}{\sqrt{6}}|1\rangle\langle 1| + \frac{1}{\sqrt{3}}|2\rangle\langle 2| + \frac{1}{\sqrt{2}}|3\rangle\langle 3|, \\ A_2 &= \frac{1}{\sqrt{6}}|1\rangle\langle 2| + \frac{1}{\sqrt{3}}|2\rangle\langle 3| + \frac{1}{\sqrt{2}}|3\rangle\langle 1|, \\ A_3 &= \frac{1}{\sqrt{6}}|1\rangle\langle 3| + \frac{1}{\sqrt{3}}|2\rangle\langle 1| + \frac{1}{\sqrt{2}}|3\rangle\langle 2|. \end{aligned}$$

For the general case $\dim \mathcal{H} = d$,

$$|\psi\rangle = \frac{1}{\sqrt{d}}|1\rangle + \frac{1}{\sqrt{d}}|2\rangle + \cdots + \frac{1}{\sqrt{d}}|d\rangle$$

and

$$K = \lambda_1|1\rangle\langle 1| + \cdots + \lambda_d|d\rangle\langle d| (\lambda_i \neq \lambda_j, i, j = 1, 2, \dots, d).$$

The construction of pure states and incoherent Kraus operators is originated from the case $\dim \mathcal{H} = 3$. Assume $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_d$. Let

$$|\phi\rangle = \sqrt{\frac{3}{2d}}|1\rangle + \sqrt{\frac{1}{2d}}|2\rangle + \sqrt{\frac{1}{d}}|3\rangle + \cdots + \sqrt{\frac{1}{d}}|d\rangle.$$

By a direct computation, we have

$$\begin{aligned} \mathcal{C}(|\phi\rangle\langle\phi|, K) - \mathcal{C}(|\psi\rangle\langle\psi|, K) \\ = \frac{\lambda_2 - \lambda_1}{4d^2} [4 \sum_{i=3}^d \lambda_i - (2d - 5)\lambda_1 - (2d - 3)\lambda_2] \\ > 0. \end{aligned}$$

Let

$$\begin{aligned} A_1 &= \sqrt{\frac{3}{2d}}|1\rangle\langle 1| + \sqrt{\frac{1}{2d}}|2\rangle\langle 2| + \sqrt{\frac{1}{d}}|3\rangle\langle 3| + \cdots \\ &\quad + \sqrt{\frac{1}{d}}|d\rangle\langle d|, \\ A_2 &= \sqrt{\frac{3}{2d}}|1\rangle\langle 2| + \sqrt{\frac{1}{2d}}|2\rangle\langle 3| + \sqrt{\frac{1}{d}}|3\rangle\langle 4| + \cdots \\ &\quad + \sqrt{\frac{1}{d}}|d-1\rangle\langle d| + \sqrt{\frac{1}{d}}|d\rangle\langle 1|, \\ A_3 &= \sqrt{\frac{3}{2d}}|1\rangle\langle 3| + \sqrt{\frac{1}{2d}}|2\rangle\langle 4| + \sqrt{\frac{1}{d}}|3\rangle\langle 5| + \cdots \\ &\quad + \sqrt{\frac{1}{d}}|d-1\rangle\langle 1| + \sqrt{\frac{1}{d}}|d\rangle\langle 2|, \\ &\dots \\ A_i &= \sqrt{\frac{3}{2d}}|1\rangle\langle m_i| + \sqrt{\frac{1}{2d}}|2\rangle\langle m_{i+1}| + \sqrt{\frac{1}{d}}|3\rangle\langle m_{i+2}| + \cdots \\ &\quad + \sqrt{\frac{1}{d}}|s\rangle\langle m_{s+i-1}| + \cdots + \sqrt{\frac{1}{d}}|d\rangle\langle m_{m_{d+i-1}}|, \\ &\dots \\ A_d &= \sqrt{\frac{3}{2d}}|1\rangle\langle d| + \sqrt{\frac{1}{2d}}|2\rangle\langle 1| + \cdots + \sqrt{\frac{1}{d}}|d\rangle\langle d-1|, \end{aligned}$$

here $m_x = x - [\frac{x-1}{d}]d$, $[\cdot]$ is the greatest integer function. It is easy to see that the Kraus operators $\{A_i\}_{i=1}^d$ define an incoherent operation. Moreover, $\sum_{i=1}^d A_i|\psi\rangle\langle\psi|A_i^\dagger = |\phi\rangle\langle\phi|$.

Generally, we can set $\lambda_{i_0}, \lambda_{j_0}, \lambda_{k_0}$ such that $\lambda_{i_0} < \lambda_{j_0} < \lambda_{k_0} < \lambda_k, k \neq i_0, j_0, k_0$. Without loss of generality, suppose $i_0 < j_0 < k_0$. Let $|\phi\rangle = \sqrt{\frac{1}{d}}|1\rangle + \dots + \sqrt{\frac{1}{d}}|i_0 - 1\rangle + \sqrt{\frac{3}{2d}}|i_0\rangle + \sqrt{\frac{1}{d}}|i_0 + 1\rangle + \dots + \sqrt{\frac{1}{d}}|j_0 - 1\rangle + \sqrt{\frac{1}{2d}}|j_0\rangle + \sqrt{\frac{1}{d}}|j_0 + 1\rangle + \dots + \sqrt{\frac{1}{d}}|k_0\rangle + \dots + \sqrt{\frac{1}{d}}|d\rangle$.

Similarly, one can construct a desired incoherent operation. \square

Conclusion—We have shown that the Wigner-Yanase skew information can increase under a class of phase-sensitive incoherent operations. It is concluded that the Wigner-Yanase skew information does not satisfy the framework of Baumgratz et al. for measure of coherence.

The Wigner-Yanase skew information is actually a good measure of asymmetry [7]. Recall that asymmetry is the properties of states to be sensitive to phase shifts along some direction. An asymmetry measure quantifies how much the symmetry in question is broken by a given state and is widely investigated in recent years [7–14].

States that are asymmetric must have coherence and a non-trivial asymmetry measure must be able to detect such coherence [7]. The author of [1] just does not distinguish between coherence and asymmetry, arguing that asymmetry is coherence in a specific basis. It is matter of taste to decide if the set of phase intensive operations or the set of incoherent operations should be considered the best pick for defining what is coherence.

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